

A Remark on the Second Neighborhood Problem

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Abstract

Seymour's second neighborhood conjecture states that every simple digraph (without digons) has a vertex whose first out-neighborhood is at most as large as its second out-neighborhood. Such a vertex is said to have the second neighborhood property (SNP). We define "good" digraphs and prove a statement that implies that every feed vertex of a tournament has the SNP. In the case of digraphs missing a matching, we exhibit a feed vertex with the SNP by refining a proof due to Fidler and Yuster and using good digraphs. Moreover, in some cases we exhibit two vertices with SNP.

1 Introduction

In this paper, a digraph D is a couple of two sets (V, D) , where $E \subseteq V \times V$. V and E are the vertex set and edge set of D and denoted by $V(D)$ and $E(D)$ respectively. An oriented graph is a digraph that contains neither loops nor digons. If $K \subseteq V(D)$ then the induced restriction of D to K is denoted by $D[K]$. As usual, $N_D^+(v)$ (resp. $N_D^-(v)$) denotes the (first) out-neighborhood (resp. in-neighborhood) of a vertex $v \in V$. $N_D^{++}(v)$ (resp. $N_D^{--}(v)$) denotes the second *out-neighborhood* (*in-neighborhood*) of v , which is the set of vertices that are at distance 2 from v (resp. to v). We also denote $d_D^+(v) = |N_D^+(v)|$, $d_D^{++}(v) = |N_D^{++}(v)|$, $d_D^-(v) = |N_D^-(v)|$ and $d_D^{--}(v) = |N_D^{--}(v)|$. We omit the subscript if the digraph is clear from the context. For short, we write $x \rightarrow y$ if the arc $(x, y) \in E$. A vertex $v \in V(D)$ is called *whole* if $d(v) := d^+(v) + d^-(v) = |V(D)| - 1$, otherwise v is non whole. A *sink* v is a vertex with $d^+(v) = 0$. For $x, y \in V(D)$, we say xy is a *missing edge* of D if neither (x, y) nor (y, x) are in $E(D)$. The *missing graph* G of D is the graph whose edges are the missing edges of D and whose vertices are the non whole vertices of D . In this case, we say that D is *missing* G . So, a tournament does not have missing edges.

A vertex v of D is said to have the *second neighborhood property* (SNP) if $d_D^+(v) \leq d_D^{++}(v)$. In 1990, Seymour conjectured the following:

Conjecture 1. (Seymour's Second Neighborhood Conjecture (SNC))[1] *Every oriented graph has a vertex with the SNP.*

In 1996, Fisher [2] solved the SNC for tournaments by using a certain probability distribution on the vertices. Another proof of Dean's conjecture was established in 2000 by Havet and Thomassé [3]. Their short proof uses a tool called median orders. Furthermore, they have proved that if a tournament has no sink vertex then there are at least two vertices with the SNP.

Let $D = (V, E)$ be a digraph (vertex) weighted by a positive real valued function $\omega : V \rightarrow \mathbb{R}_+$. The couple (D, ω) (or simply D) is called a *weighted digraph*. The weight of an arc $e = (x, y)$ is $\omega(e) := \omega(x) \cdot \omega(y)$. The weight of a set of vertices (resp. edges) is the sum of the weights of its members. We say that a vertex v has the *weighted SNP* if $\omega(N^+(v)) \leq \omega(N^{++}(v))$. It is known that the SNC is equivalent to its weighted version: *Every weighted oriented graph has a vertex with the weighted SNP.*

Let $L = v_1 v_2 \dots v_n$ be an ordering of the vertices of a weighted digraph (D, ω) . An arc $e = (v_i, v_j)$ is *forward* with respect to L if $i < j$. Otherwise e is a *backward* arc. A *weighted median order* $L = v_1 v_2 \dots v_n$ of D is an order of the vertices of D that maximizes the weight of the set of forward arcs of D , i.e., the set $\{(v_i, v_j) \in E(D); i < j\}$. In other words, $L = v_1 v_2 \dots v_n$ is a weighted median order of D if $\omega(L) = \max\{\omega(L'); L' \text{ is an ordering of the vertices of } D\}$. In fact, the weighted median order L satisfies the *feedback property*: For all $1 \leq i \leq j \leq n$:

$$\omega(N_{D[i,j]}^+(v_i)) \geq \omega(N_{D[i,j]}^-(v_i))$$

and

$$\omega(N_{D[i,j]}^-(v_j)) \geq \omega(N_{D[i,j]}^+(v_j))$$

where $[i, j] := \{v_i, v_{i+1}, \dots, v_j\}$.

Indeed, suppose to the contrary that $\omega(N_{D[i,j]}^+(v_i)) < \omega(N_{D[i,j]}^-(v_i))$. Consider the order $L' = v_1 \dots v_{i-1} v_{i+1} \dots v_j v_i v_{j+1} \dots v_n$ obtained from L by inserting v_i just after v_j . Then we have:

$$\begin{aligned} \omega(L') &= \omega(L) + \omega(\{(v_k, v_i) \in E(D); i \leq k \leq j\}) - \omega(\{(v_i, v_k) \in E(D); i \leq k \leq j\}) \\ &= \omega(L) + \omega(v_i) \cdot \omega(N_{D[i,j]}^-(v_i)) - \omega(v_i) \cdot \omega(N_{D[i,j]}^+(v_i)) \end{aligned}$$

$$= \omega(L) + \omega(v_i) \cdot (\omega(N_{D[i,j]}^-(v_i)) - \omega(N_{D[i,j]}^+(v_j))) > \omega(L),$$

which contradicts the maximality of $\omega(L)$.

It is also known that if we reverse the orientation of a backward arc $e = (v_i, v_j)$ of D with respect to L , then L is again a weighted median order of the new weighted digraph $D' = D - (v_i, v_j) + (v_j, v_i)$.

When $\omega = 1$, we obtain the definition of median orders of a digraph ([3, 4]).

Let $L = v_1 v_2 \dots v_n$ be a weighted median order. Among the vertices not in $N^+(v_n)$ two types are distinguished: A vertex v_j is *good* if there is $i \leq j$ such that $v_n \rightarrow v_i \rightarrow v_j$, otherwise v_j is a *bad vertex*. The set of good vertices of L is denoted by G_L^D [3] (or G_L if there is no confusion). Clearly, $G_L \subseteq N^{++}(v_n)$. The last vertex v_n is called a feed vertex of (D, ω) .

In 2007, Fidler and Yuster [4] proved that SNC holds for oriented graphs missing a matching. They have used median orders and another tool called the dependency digraph. However, their proof does not guarantee that the vertex found to have the SNP is a feed vertex.

In 2012, Ghazal also used the notion of weighted median order to prove the weighted SNC for digraphs missing a generalized star. As a corollary, the weighted version holds for digraphs missing a star, complete graph or a sun [5]. He also used the dependency digraph to prove SNC for other classes of oriented graphs [6].

We say that a missing edge $x_1 y_1$ *loses to* a missing edge $x_2 y_2$ if: $x_1 \rightarrow x_2$, $y_2 \notin N^+(x_1) \cup N^{++}(x_1)$, $y_1 \rightarrow y_2$ and $x_2 \notin N^+(y_1) \cup N^{++}(y_1)$. The *dependency digraph* Δ of D is defined as follows: Its vertex set consists of all the missing edges and $(ab, cd) \in E(\Delta)$ if ab loses to cd [4, 6]. Note that Δ may contain digons.

Definition 1. [5] In a digraph D , a missing edge ab is called a *good missing edge* if:

- (i) $(\forall v \in V \setminus \{a, b\})[(v \rightarrow a) \Rightarrow (b \in N^+(v) \cup N^{++}(v))]$ or
- (ii) $(\forall v \in V \setminus \{a, b\})[(v \rightarrow b) \Rightarrow (a \in N^+(v) \cup N^{++}(v))]$.

If ab satisfies (i) we say that (a, b) is a *convenient orientation* of ab .

If ab satisfies (ii) we say that (b, a) is a *convenient orientation* of ab .

We will need the following observation:

Lemma 1. [4] *Let D be an oriented graph and let Δ denote its dependency digraph. A missing edge ab is good if and only if its in-degree in Δ is zero.*

In the next section, we will define good median orders and good digraphs and prove a statement which implies that every feed vertex of a weighted tournament has the weighted SNP. In the last section, we refine the proof of Fidler and Yuster and use good median orders to exhibit a feed vertex with the SNP in the case of oriented graphs missing a matching.

2 Good median orders

Let D be a (weighted) digraph and let Δ denote its dependency digraph. Let C be a connected component of Δ . Set $K(C) = \{u \in V(D); \text{there is a vertex } v \text{ of } D \text{ such that } uv \text{ is a missing edge and belongs to } C\}$. The *interval graph of D* , denoted by \mathcal{I}_D is defined as follows. Its vertex set consists of the connected components of Δ and two vertices C_1 and C_2 are adjacent if $K(C_1) \cap K(C_2) \neq \emptyset$. So \mathcal{I}_D is the intersection graph of the family $\{K(C); C \text{ is a connected component of } \Delta\}$. Let ξ be a connected component of \mathcal{I}_D . We set $K(\xi) = \cup_{C \in \xi} K(C)$. Clearly, if uv is a missing edge in D then there is a unique connected component ξ of \mathcal{I}_D such that u and v belong to $K(\xi)$. For $f \in V(D)$, we set $J(f) = \{f\}$ if f is a whole vertex, otherwise $J(f) = K(\xi)$, where ξ is the unique connected component of \mathcal{I}_D such that $f \in K(\xi)$. Clearly, if $x \in J(f)$ then $J(f) = J(x)$ and if $x \notin J(f)$ then x is adjacent to every vertex in $J(f)$.

Let $L = x_1 \cdots x_n$ be a (weighted) median order of a digraph D . For $i < j$, the sets $[i, j] := [x_i, x_j] := \{x_i, x_{i+1}, \dots, x_j\}$ and $]i, j[:= [i, j] \setminus \{x_i, x_j\}$ are called *intervals* of L . We recall that $K \subseteq V(D)$ is an *interval of D* if for every $u, v \in K$ we have: $N^+(u) \setminus K = N^+(v) \setminus K$ and $N^-(u) \setminus K = N^-(v) \setminus K$. The following shows a relation between the intervals of D and the intervals of L .

Proposition 1. *Let $\mathcal{I} = \{I_1, \dots, I_r\}$ be a set of pairwise disjoint intervals of D . Then for every weighted median order L of D , there is a weighted median order L' of D such that: L and L' have the same feed vertex and every interval in \mathcal{I} is an interval of L' .*

Proof. Let $L = x_1 x_2 \dots x_n$ be a weighted median order of a weighted digraph (D, ω) and let $\mathcal{I} = \{I_1, \dots, I_r\}$ be a set of pairwise disjoint intervals of D . We will use

the feedback property to prove it. Suppose $a, b \in I_1$ with $a = x_i$, $b = x_j$, $i < j$ and $[x_i, x_j] \cap I_1 = \{x_i, x_j\}$. Since I_1 is an interval of D , we have $N_{[i,j]}^+(x_i) = N_{[i,j]}^+(x_j)$ and $N_{[i,j]}^-(x_i) = N_{[i,j]}^-(x_j)$. So, $\omega(N_{[i,j]}^-(x_i)) \leq \omega(N_{[i,j]}^+(x_i)) = N_{[i,j]}^+(x_j) \leq \omega(N_{[i,j]}^-(x_j)) = \omega(N_{[i,j]}^-(x_i))$, where the two inequalities are by the feedback property. Whence, all the quantities in the previous statement are equal. In particular, $\omega(N_{[i,j]}^+(x_i)) = \omega(N_{[i,j]}^-(x_i))$. Let L_1 be the enumeration $x_1 \dots x_{i-1} x_{i+1} \dots x_{j-1} x_i x_j x_{j+1} \dots x_n$. Then $\omega(L_1) = \omega(L) + \omega(N_{[i,j]}^-(x_i)) - \omega(N_{[i,j]}^+(x_i)) = \omega(L)$. Thus, L_1 is a weighted median order of D . By successively repeating this argument, we obtain a weighted median order in which I_1 is an interval of L . Again, by successively repeating the argument for each $I \in \mathcal{I}$, we obtain the desired order. \square

We say that D is *good digraph* if the sets $K(\xi)$'s are intervals of D . By the previous proposition, every good digraph has a (weighted) median order L such that the $K(\xi)$'s form intervals of L . Such an enumeration is called a *good (weighted) median order* of the good digraph D .

Theorem 1. *Let (D, ω) be a good weighted oriented graph and let L be a good weighted median order of (D, ω) , with feed vertex say f . Then for every $x \in J(f)$, we have $\omega(N^+(x) \setminus J(f)) \leq \omega(G_L \setminus J(f))$. So if x has the weighted SNP in $(D[J(f)], \omega)$, then it has the weighted SNP in D .*

Proof. The proof is by induction n , the number of vertices of D . It is trivial for $n = 1$. Let $L = x_1 \dots x_n$ be a good weighted median order of (D, ω) . Set $f = x_n$, $J(f) = [x_t, x_n]$, $L_1 = x_1 \dots x_t$ and $D_1 = D[x_1, x_t]$. Then (D_1, ω) is a good weighted oriented graph and L_1 is a good weighted median order of (D_1, ω) in which $J(x_t) = \{x_t\}$. Suppose that $t < n$. Then by the induction hypothesis, $\omega(N_{D_1}^+(x_t)) \leq \omega(G_{L_1})$. However, $J(f)$ is an interval of D , then for every $x \in J(f)$, we have $\omega(N^+(x) \setminus J(f)) = \omega(N^+(x_t) \setminus J(f)) = \omega(N_{D_1}^+(x_t)) \leq \omega(G_{L_1}) = \omega(G_L \setminus J(f))$. Now suppose that $t = n$. If L does not have any bad vertex then $N^-(x_n) = G_L$. Whence, $\omega(N^+(x_n)) \leq \omega(N^-(x_n)) = \omega(G_L)$ where the inequality is by the feedback property. Now suppose that L has a bad vertex and let i be the smallest such that x_i is bad. Since $J(x_i)$ is an interval of D and L , then every vertex in $J(x_i)$ is bad and thus $J(x_i) = [x_i, x_p]$ for some $p < n$. For $j < i$, x_j is either an out-neighbor of x_n or a good vertex, by definition of i . Moreover, if $x_j \in N^+(x_n)$ then $x_j \in N^+(x_i)$. So $N^+(x_n) \cap [1, i] \subseteq N^+(x_i) \cap [1, i]$. Equivalently, $N^-(x_i) \cap [1, i] \subseteq G_L \cap [1, i]$. Therefore, $\omega(N^+(x_n) \cap [1, i]) \leq \omega(N^+(x_i) \cap [1, i]) \leq \omega(N^-(x_i) \cap [1, i]) \leq \omega(G_L \cap [1, i])$, where the second inequality is by the feedback

property. Now $L' = x_{p+1} \dots x_n$ is good also. By induction, $\omega(N^+(x_n) \cap [p+1, n]) \leq \omega(G_{L'})$. Note that $G_{L'} \subseteq G_L \cap [p+1, n]$. Whence $\omega(N^+(x_n)) = \omega(N^+(x_n) \cap [1, i]) + \omega(N^+(x_n) \cap [p+1, n]) \leq \omega(G_L \cap [1, i]) + \omega(G_L \cap [p+1, n]) = \omega(G_L)$. The second part of the statement is obvious. \square

Since every (weighted) tournament is a good (weighted) oriented graph, we obtain the following two results.

Corollary 1. ([4]) *Let L be a weighted median order of a weighted tournament (T, ω) with feed vertex say f . Then $\omega(N^+(f)) \leq \omega(G_L)$.*

Corollary 2. ([3]) *Let L be a median order of a tournament with feed vertex say f . Then $|N^+(f)| \leq |G_L|$.*

Let L be a good weighted median order of a good oriented graph D and let f denote its feed vertex. By theorem 1, for every $x \in J(f)$, $\omega(N^+(x) \setminus J(f)) \leq \omega(G_L \setminus J(f))$. Let b_1, \dots, b_r denote the bad vertices of L not in $J(f)$ and v_1, \dots, v_s denote the non bad vertices of L not in $J(f)$, both enumerated in increasing order with respect to their index in L .

If $\omega(N^+(f) \setminus J(f)) < \omega(G_L \setminus J(f))$, we set $Sed(L) = L$. If $\omega(N^+(f) \setminus J(f)) = \omega(G_L \setminus J(f))$, we set $sed(L) = b_1 \dots b_r J(f) v_1 \dots v_s$. This new order is called the *sedimentation of L* .

Lemma 2. *Let L be a good weighted median order of a good weighted oriented graph (D, ω) . Then $Sed(L)$ is a good weighted median order of (D, ω) .*

Proof. Let $L = x_1 \dots x_n$ be a good weighted local median order of (D, ω) . If $Sed(L) = L$, there is nothing to prove. Otherwise, we may assume that $\omega(N^+(x_n) \setminus J(x_n)) = \omega(G_L \setminus J(x_n))$. The proof is by induction on r the number of bad vertices not in $J(x_n)$. Set $J(x_n) = [x_t, x_n]$. If $r = 0$ then we have $N^-(x_n) \setminus J(x_n) = G_L \setminus J(x_n)$. Whence, $\omega(N^+(x_n) \setminus J(x_n)) = \omega(G_L \setminus J(x_n)) = \omega(N^-(x_n) \setminus J(x_n))$. Thus, $Sed(L) = J(x_n)x_1 \dots x_{t-1}$ is a good weighted median order. Now suppose that $r > 0$ and let i be the smallest such that $x_i \notin J(x_n)$ and is bad. As in the proof of theorem 1, $J(x_i) = [x_i, x_p]$ for some $p < n$, $\omega(N^+(x_n) \cap [1, i]) \leq \omega(N^+(x_i) \cap [1, i]) \leq \omega(N^-(x_i) \cap [1, i]) \leq \omega(G_L \cap [1, i])$ and $\omega(N^+(x_n) \cap [p+1, t-1]) \leq \omega(G_L \cap [p+1, t-1])$. However, $\omega(N^+(x_n) \setminus J(x_n)) = \omega(G_L \setminus J(x_n))$, then the previous inequalities are equalities. In particular, $\omega(N^+(x_i) \cap [1, i]) = \omega(N^-(x_i) \cap [1, i])$. Since $J(x_i)$ is an interval of L and D , then for every $x \in J(x_i)$ we have $\omega(N^+(x) \cap [1, i]) = \omega(N^-(x) \cap [1, i])$. Thus $J(x_i)x_1 \dots x_{i-1}x_{p+1} \dots x_n$ is a good weighted median order.

To conclude, apply the induction hypothesis to the good weighted median order $x_1 \dots x_{i-1} x_{p+1} \dots x_n$.

□

Define now inductively $Sed^0(L) = L$ and $Sed^{q+1}(L) = Sed(Sed^q(L))$. If the process reaches a rank q such that $Sed^q(L) = y_1 \dots y_n$ and $\omega(N^+(y_n) \setminus J(y_n)) < \omega(G_{Sed^q(L)} \setminus J(y_n))$, call the order L *stable*. Otherwise call L *periodic*. These new order are used by Havet and Thomassé to exhibit a second vertex with the SNP in tournaments that do not have any sink. We will use them for the same purpose but for other classes of oriented graphs.

3 Case of oriented graph missing a matching

In this section, D is an oriented graph missing a matching and Δ denotes its dependency digraph. We begin by the following lemma:

Lemma 3. [4] *The maximum out-degree of Δ is one and the maximum in-degree of Δ is one. Thus Δ is composed of vertex disjoint directed paths and directed cycles.*

Proof. Assume that $a_1 b_1$ loses to $a_2 b_2$ and $a_1 b_1$ loses to $a'_2 b'_2$, with $a_1 \rightarrow a_2$ and $a_1 \rightarrow a'_2$. The edge $a'_2 b_2$ is not a missing edge of D . If $a'_2 \rightarrow b_2$ then $b_1 \rightarrow a'_2 \rightarrow b_2$, a contradiction. If $b_2 \rightarrow a'_2$ then $b_1 \rightarrow b_2 \rightarrow a'_2$, a contradiction. Thus, the maximum out-degree of Δ is one. Similarly, the maximum in-degree is one. □

In the following, $C = a_1 b_1, \dots, a_k b_k$ denotes a directed cycle of Δ , namely $a_i \rightarrow a_{i+1}$, $b_{i+1} \notin N^{++}(a_i) \cup N^+(a_i)$, $b_i \rightarrow b_{i+1}$ and $a_{i+1} \notin N^{++}(b_i) \cup N^+(b_i)$, for all $i < k$. In [4], it is proved that $D[K(C)]$ has a vertex with the SNP. Here we prove that every vertex of $K(C)$ has the SNP in $D[K(C)]$.

Lemma 4. ([4]) *If k is odd then $a_k \rightarrow a_1$, $b_1 \notin N^{++}(a_k) \cup N^+(a_k)$, $b_k \rightarrow b_1$ and $a_1 \notin N^{++}(b_k) \cup N^+(b_k)$. If k is even then $a_k \rightarrow b_1$, $a_1 \notin N^{++}(a_k) \cup N^+(a_k)$, $b_k \rightarrow a_1$ and $b_1 \notin N^{++}(b_k) \cup N^+(b_k)$.*

Lemma 5. [4] *$K(C)$ is an interval of D .*

Proof. Let $f \notin K(C)$. Then f is adjacent to every vertex in $K(C)$. If $a_1 \rightarrow f$ then $b_2 \rightarrow f$, since otherwise $b_2 \in N^{++}(a_1) \cup N^+(a_1)$ which is a contradiction. So $N^+(a_1) \setminus K(C) \subseteq N^+(b_2) \setminus K(C)$. Applying this to every losing relation of C yields $N^+(a_1) \setminus K(C) \subseteq N^+(b_2) \setminus K(C) \subseteq N^+(a_3) \setminus K(C) \dots \subseteq N^+(b_k) \setminus K(C) \subseteq$

$N^+(b_1) \setminus K(C) \subseteq N^+(a_2) \setminus K(C) \dots \subseteq N^+(a_k) \setminus K(C) \subseteq N^+(a_1) \setminus K(C)$ if k is even. So these inclusion are equalities. An analogous argument proves the same result for odd cycles. \square

Lemma 6. *In $D[K(C)]$ we have:*

If k is odd then:

$$N^+(a_1) = N^-(b_1) = \{a_2, b_3, \dots, a_{k-1}, b_k\}$$

$$N^-(a_1) = N^+(b_1) = \{b_2, a_3, \dots, b_{k-1}, a_k\},$$

If k is even then:

$$N^+(a_1) = N^-(b_1) = \{a_2, b_3, \dots, b_{k-1}, a_k\}$$

$$N^-(a_1) = N^+(b_1) = \{b_2, a_3, \dots, a_{k-1}, b_k\}.$$

Proof. Suppose that k is odd. Set $K := K(C)$. Then $b_k \in N_{D[K]}^+(a_1)$ by lemma 4. Since $a_{k-1}b_{k-1}$ loses to a_k, b_k and $(a_1, b_k) \in E(D)$ then $(a_1, a_{k-1}) \in E(D)$ and so $a_{k-1} \in N_{D[K]}^+(a_1)$, since otherwise $(a_{k-1}, a_1) \in E(D)$ and so $b_k \in N_{D[K]}^{++}(a_{k-1})$, which is a contradiction to the definition of the losing relation $a_{k-1}b_{k-1} \rightarrow a_kb_k$. And so on $b_{k-2}, a_{k-3}, \dots, b_3, a_2 \in N_{D[K]}^+(a_1)$. Again, since a_1b_1 loses to a_2, b_2 then $b_2 \in N_{D[K]}^-(a_1)$. Since a_2b_2 loses to a_3, b_3 and $(b_2, a_1) \in E(D)$ then $(a_3, a_1) \in E(D)$ and so $a_3 \in N_{D[K(C)]}^-(a_1)$. And so on, $b_4, a_5, \dots, b_{k-1}, a_k \in N_{D[K]}^-(a_1)$. We use the same argument for finding $N_{D[K]}^+(b_1)$ and $N_{D[K]}^-(b_1)$. Also we use the same argument when k is even. \square

Lemma 7. *In $D[K(C)]$ we have: $N^+(a_i) = N^-(b_i)$, $N^-(a_i) = N^+(b_i)$, $N^{++}(a_i) = N^-(a_i) \cup \{b_i\} \setminus \{b_{i+1}\}$ and $N^{++}(b_i) = N^-(b_i) \cup \{a_i\} \setminus \{a_{i+1}\}$ for all $i = 1, \dots, k$ where $a_{k+1} := a_1$, $b_{k+1} := b_1$ if k is odd and $a_{k+1} := b_1$, $b_{k+1} := a_1$ if k is even.*

Proof. The first part is due to the previous lemma and the symmetry in these cycles. For the second part it is enough to prove it for $i = 1$ and a_1 . Suppose first that k is odd. By definition of losing relation between a_1b_1 and a_2b_2 we have $b_2 \notin N^{++}(a_1) \cup N^+(a_1)$. Moreover $a_1 \rightarrow a_2 \rightarrow b_1$, whence $b_1 \in N^{++}(a_1)$. Note that for $i = 1, \dots, k-1$, $a_i \rightarrow a_{i+1}$ and $b_i \rightarrow b_{i+1}$. Combining this with the previous lemma we find that $N^{++}(a_1) = N^-(a_1) \cup \{b_1\} \setminus \{b_2\}$. Similar argument is used when k is even. \square

So we have:

Lemma 8. $d^{++}(v) = d^+(v) = d^-(v) = k - 1$ for all $v \in K(C)$.

Let $P = a_1b_1, a_2b_2, \dots, a_kb_k$ be a connected component of Δ , which is also a maximal path in Δ , namely $a_i \rightarrow a_{i+1}, b_i \rightarrow b_{i+1}$ for $i = 1, \dots, k - 1$. Since a_1b_1 is a good edge then (a_1, b_1) or (b_1, a_1) is a convenient orientation. If (a_1, b_1) is a convenient orientation, then we orient (a_i, b_i) for $i = 1, \dots, k$. Otherwise, we orient a_ib_i as (b_i, a_i) . We do this for every such a path of Δ . Denote the set of these new arcs by F . Set $D' = D + F$.

Since we have oriented all the missing edges of D that form the connected components of Δ that are paths, then they are no longer missing edges of D' and thus, the dependency digraph of D' is composed of only directed cycles. Then by lemma 5 we have:

Lemma 9. D' is a good digraph.

Now, we are ready to prove the following statement:

Theorem 2. Every feed vertex of D' has the SNP in D and D' .

Proof. Let L be a good median order of D' and let f denote its feed vertex. We have $|N_{D'}^+(f) \setminus J(f)| \leq |G_L^{D'} \setminus J(f)|$ by theorem 1.

Suppose that f is not incident to any new arc of F . Then $J(f) = \{f\}$ or $J(f) = K(C)$ (in D and D') for some cycle C of Δ , $N_{D'}^+(f) = N^+(f)$ and f has the SNP in $D[J(f)]$. Let $y \in N_{D'}^{++}(f) \setminus J(f)$. There is a vertex x such that $f \rightarrow x \rightarrow y \rightarrow f$ in D' . Note that the arcs (f, x) and (y, f) are in D . If $(x, y) \in D$ or is a convenient orientation then $y \in N^{++}(f)$. Otherwise, there is a missing edge rs that loses to xy , namely $s \rightarrow y$ and $x \notin N^{++}(s) \cup N^+(s)$. But fs is not a missing edge then we must have $(f, s) \in D$. Thus $y \in N^{++}(f)$. Hence $N_{D'}^{++}(f) \setminus J(f) \subseteq N^{++}(f) \setminus J(f)$. Thus $|N^+(f)| = |N_{D'}^+(f)| = |N_{D'}^+(f) \setminus J(f)| + |N_{D'}^+(f) \cap J(f)| \leq |G_L^{D'} \setminus J(f)| + |N_{D'}^{++}(f) \cap J(f)| \leq |N^{++}(f) \setminus J(f)| + |N_{D[J(f)]}^{++}(f)| = |N^{++}(f)|$.

Suppose that F is incident to a new arc of F . Then there is a path $P = a_1b_1, a_2b_2, \dots, a_kb_k$ in Δ , which is also a connected component Δ , namely $a_t \rightarrow a_{t+1}, b_t \rightarrow b_{t+1}$ for $t = 1, \dots, k - 1$, such that $f = a_i$ or $f = b_i$. We may suppose without loss of generality that $(a_t, b_t) \in D'$, $\forall t \in \{1, \dots, k\}$. Suppose first that

$f = a_i$ and $i < k$. Then f gains only b_i as a first out-neighbor and b_{i+1} as a second out-neighbor. Indeed, let $y \in N_{D'}^{++}(f) \setminus \{b_{i+1}\}$. There is a vertex x such that $f \rightarrow x \rightarrow y \rightarrow f$ in D' . Suppose that $b_i \neq x$. Note that the arcs (f, x) and (y, f) are in D . If $(x, y) \in D$ or is a convenient orientation then $y \in N^{++}(f)$. Otherwise, there is a missing edge rs that loses to xy , namely $s \rightarrow y$ and $x \notin N^{++}(s) \cup N^+(s)$. But fs is not a missing edge then we must have $(f, s) \in D$. Thus $y \in N^{++}(f)$. Suppose that $b_i = x$. Since $b_i \rightarrow y$, $a_{i+1} \notin N^{++}(b_i) \cup N^+(b_i)$ and $a_{i+1}y$ is not a missing edge, then we must have $(y, a_{i+1}) \in D$. Thus $f \rightarrow a_{i+1} \rightarrow y$ in D and $y \in N^{++}(f)$. Hence $N_{D'}^{++}(f) \setminus \{b_{i+1}\} \subseteq N^{++}(f)$. Note that $J(f) = \{f\}$ in D' . Combining this with theorem 1, we get $|N^+(f)| = |N_{D'}^+(f)| - 1 \leq |N_{D'}^{++}(f)| - 1 \leq |N^{++}(f)|$. Now suppose that $f = a_k$. We reorient the missing edge $a_k b_k$ as (b_k, a_k) and let D'' denote the new oriented graph. Then L is a good median order of the good oriented graph D'' , $N_{D''}^+(f) = N^+(f)$, $J(f) = \{f\}$ in D'' , and f has the SNP in D'' . Let $y \in N_{D''}^{++}(f)$. There is a vertex x such that $f \rightarrow x \rightarrow y \rightarrow f$ in D'' . Note that the arcs (f, x) and (y, f) are in D . If $(x, y) \in D$ or is a convenient orientation then $y \in N^{++}(f)$. Otherwise, there is a missing edge rs that loses to xy , namely $s \rightarrow y$ and $x \notin N^{++}(s) \cup N^+(s)$. But fs is not a missing edge then we must have $(f, s) \in D$. Thus $y \in N^{++}(f)$ and $N_{D''}^{++}(f) \subseteq N^{++}(f)$. Thus f has the SNP in D . Finally, suppose that $f = b_i$. We use the same argument of the case $f = a_k$ to prove that f has the SNP in D . \square

We note that our method guarantees that the vertex f found with the SNP is a feed vertex of some digraph containing D . This is not guaranteed by the proof presented in [4]. Recall that F is the set of the new arcs added to D to obtain the good oriented graph D' . So if $F = \emptyset$ then D is a good oriented graph.

Theorem 3. *Let D be an oriented graph missing a matching and suppose that $F = \emptyset$. If D has no sink vertex then it has at least two vertices with the SNP.*

Proof. Consider a good median order $L = x_1 \dots x_n$ of D . If $J(x_n) = K(C)$ for some directed cycle C of Δ then by lemma 1 and lemma 8 the result holds. Otherwise, x_n is a whole vertex (i.e. $J(x_n) = \{x_n\}$). By lemma 1, x_n has the SNP in D . So we need to find another vertex with SNP. Consider the good median order $L' = x_1 \dots x_{n-1}$. Suppose first that L' is stable. There is q for which $Sed^q(L') = y_1 \dots y_{n-1}$ and $|N^+(y_{n-1}) \setminus J(y_{n-1})| < |G_{Sed^q(L')} \setminus J(y_{n-1})|$. Note that $y_1 \dots y_{n-1} x_n$ is also a good median order of D . By lemma 1 and lemma 8, $y := y_{n-1}$ has the SNP in $D[y_1, y_{n-1}]$. So $|N^+(y)| = |N_{D[y_1, y_{n-1}]}^+(y)| + 1 \leq |G_{Sed^q(L')}| \leq |N^{++}(y)|$. Now suppose that L' is periodic. Since D has no sink then x_n has an out-neighbor x_j .

Choose j to be the greatest (so that it is the last vertex of its corresponding interval). Note that for every q , x_n is an out-neighbor of the feed vertex of $Sed^q(L')$. So x_j is not the feed vertex of any $Sed^q(L')$. Since L' is periodic, x_j must be a bad vertex of $Sed^q(L')$ for some integer q , otherwise the index of x_j would always increase during the sedimentation process. Let q be such an integer. Set $Sed^q(L') = y_1 \dots y_{n-1}$. Lemma 8 and lemma 1 guarantee that the vertex $y := y_{n-1}$ with the SNP in $D[y_1, y_{n-1}]$. Note that $y \rightarrow x_n \rightarrow x_j$ and $G_{Sed^q(L')} \cup \{x_j\} \subseteq N^{++}(y)$. So $|N^+(y)| = |N_{D[y_1, y_{n-1}]}^+(y)| + 1 = |G_{Sed^q(L')} + 1| = |G_{Sed^q(L')} \cup \{x_j\}| \leq |N^{++}(y)|$. \square

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